Units for other quantities will be introduced as needed throughtout the book.

An important part of learning about science and technology is learning how to calculate physical quantities in real-life situations. This is useful in everything from estimating your home heating bill to understanding how computer circuits work. Here we will study a systematic method for calculating quantities involving units.

Consider a car traveling with speed equal to S. The formula for the distance (D) traveled in a certain time (t) by the car is: $D = S \cdot t$, where the dot means "times." Common sense tells us that if the car's speed is 100 kilometers per hour (100 km/hr), and the car travels for 2 hours, the distance it covers is 200 km. That is,

$$D = (100 \text{ km/hr}) \cdot (2 \text{ hr}) = 200 \text{ km}$$

There are several equivalent ways to write 100 kilometers per hour:

$$100 \text{ km per hr} = 100 \text{ km/hr} = 100 \frac{\text{km}}{\text{hr}} = \frac{100 \text{ km}}{\text{hr}}$$

The word "per" acts like division (÷) for the units.

Let us analyze the distance calculation in more detail:

$$D = S \cdot t = (100 \text{ km per hr}) \cdot (2\text{hr}) = \frac{100 \text{ km}}{\cancel{\text{hr}}} \cdot 2 \ \cancel{\text{hr}} = 200 \text{ km}$$

The hr unit appears both in the numerator and the denominator and it therefore cancels, just as in ordinary fractions involving numbers or variables; for example,

$$\left(\frac{a}{b}\right) \cdot b = a$$

Notice also that we must use values of the speed and time that are compatible, so that the unit of time (hours) will cancel properly, as in the above example. If instead we were to give the time in minutes (120 min) and put this into the formula, we would get

$$D = S \cdot t = \frac{100 \text{ km}}{\text{hr}} \cdot 120 \text{ min} = 1200 \text{ km} \frac{\text{min}}{\text{hr}}$$

Written this way, the minutes and hours units do not cancel, and we get a result that is hard to interpret. To remedy this, we need to use the following method.

2.5.2 The Method of Conversion Factors

In the preceding example, we were left with the awkward units min/hr, which we want to eliminate. We know that there are 60 minutes per hour; this means that there are 1/60 hour per minute. Another way to say this is that 60 minutes equals 1 hour, or 60 min = 1 hr. This means that 60 min divided by 1 hr equals 1.

$$\left(\frac{60 \text{ min}}{1 \text{ hr}}\right)_{\text{CF}} = 1$$

In this equation, the "1" on the right-hand side has no units. This quantity is called a ***conversion factor***, and we write "CF" near the bottom of the bracket to remind us. It is also true that 1 hr divided by 60 minutes equals 1:

$$\left(\frac{1 \text{ hr}}{60 \text{ min}}\right)_{\text{CF}} = 1$$

We can multiply any quantity by any conversion factor without changing the quantity's value. Consider a simple example: What does 3 hours equal in minutes?

$$3\text{ hr} = 3\text{ hr}\cdot(1) = 3\text{ hr}\cdot\left(\frac{60\text{ min}}{1\text{ hr}}\right)_{\text{CF}} = 3\ \cancel{\text{hr}}\cdot\frac{60\text{ min}}{\cancel{\text{hr}}} = 180\text{ min}$$

The numerical value (3) has been changed (to 180), but the quantity of time has not changed. We can also illustrate this method in the opposite direction:

$$180\text{ min} = 180\text{ min}\cdot(1) = 180\text{ min}\cdot\left(\frac{1\text{ hr}}{60\text{ min}}\right)_{\text{CF}} = 180\,\cancel{\text{min}}\cdot\frac{1\text{ hr}}{60\,\cancel{\text{min}}} = 3\text{hr}$$

When doing such a calculation on paper, it is most efficient not to write every step shown above. First write "180 min," then next to it write the conversion factor, then cancel units, then evaluate the numerical part, and write the answer. The final product looks like:

$$180\,\cancel{\text{min}}\cdot\left(\frac{1\text{ hr}}{60\ \cancel{\text{min}}}\right)_{\text{CF}} = 3\text{ hr}$$

It is good to always put brackets around conversion factors to remind you of the method.

Now we can apply the conversion factor method to the above example for the distance traveled in 120 minutes by a car moving with speed 100 kilometers per hour:

$$D = \frac{100\text{ km}}{\text{hr}}\cdot 120\text{ min} = \frac{100\text{ km}}{\cancel{\text{hr}}}\cdot 120\ \cancel{\text{min}}\cdot\left(\frac{1\ \cancel{\text{hr}}}{60\ \cancel{\text{min}}}\right)_{\text{CF}} = 200\text{ km}$$

We multiplied by the conversion factor (1 hr/60 min), canceled the minutes units with each other, and then canceled the hour units with each other, leaving kilometers.

QUICK QUESTION 2.2

Use a conversion factor method to convert 3700 meters into kilometers. Recall that there are 1000 meters in a kilometer.

There is another way to do such a calculation that some people find easier to understand, although it takes more steps. The speed (*S*) is given as 100 kilometers per hour, and the time (*t*) traveled is 120 minutes. The formula for distance is $D = S\cdot t$. The problem here is that the units are not consistently given in the speed and in the time. First, convert the time from minutes to units of hours before inserting it into the formula. This is done by multiplying 120 min by the proper conversion factor to give

$$120\text{ min} = 120\ \cancel{\text{min}}\cdot\left(\frac{1\text{ hr}}{60\ \cancel{\text{min}}}\right)_{\text{CF}} = 2\text{ hr}$$

After converting the time into units of hours, insert it into the distance formula:

$$D = S\cdot t = \frac{100\text{ km}}{\cancel{\text{hr}}}\cdot 2\ \cancel{\text{hr}} = 200\text{ km}$$

Finally, note that you can also divide by conversion factors instead of multiplying by them. Because a conversion factor equals 1, either multiplication or division by 1 will leave the original quantity unchanged, except for its units. For example,

$$180\text{ min} = \frac{180\text{ min}}{\left(\frac{60\text{ min}}{1\text{ hr}}\right)_{\text{CF}}} = 180\ \cancel{\text{min}}\cdot\left(\frac{1\text{ hr}}{60\ \cancel{\text{min}}}\right)_{\text{CF}} = 3\text{ hr}$$

Dividing by a fraction is equivalent to multiplying by the fraction's inverse. For example,

$$\frac{a}{\left(\frac{b}{c}\right)} = a \cdot \left(\frac{c}{b}\right),$$

and therefore,

$$\frac{a}{\left(\frac{a}{c}\right)} = \not{a} \cdot \left(\frac{c}{\not{a}}\right) = c$$

IN-DEPTH LOOK 2.1: USING CONVERSION FACTORS

We can make up a general example of using conversion factors by using arbitrary names for the units. For fun, I will call these *whatnot* and *whosis*. Let us say we begin with a value of 125 *whatnot*. How many *whosis* does this equal? To answer, we need to know how many *whosis* make up a single *whatnot*, or vice versa. Let us use an example in which 1 *whosis* is equivalent to 25 *whatnot*. So, we can write

1 *whosis* = 25 *whatnot*, or 1 *whatnot* = (1/25) *whosis*

That is, 1 *whatnot* equals one twenty-fifth of a *whosis*. Because 1/25 = 0.04, we could also write this as 1 *whatnot* = 0.04 *whosis*. We can now create a conversion factor by noting that

$$\left(\frac{1\,whosis}{25\,whatnot}\right)_{CF} = 1$$

We can use this conversion factor to convert our starting number:

$$125\,whatnot \times \left(\frac{1\,whosis}{25\,whatnot}\right)_{CF} = 5\,whosis$$

We can also have an example that goes in the opposite direction. If we start out with 3 *whosis*, how many *whatnots* is this? We multiply by the other conversion factor:

$$3\,whosis \times \left(\frac{25\,whatnot}{1\,whosis}\right)_{CF} = 75\,whatnot$$

Now that you have struggled through this example using the silly terms *whatnot* and *whosis*, go back and reread it substituting *cents* for *whatnot*, and *quarters* for *whosis*. You will see that it makes a lot of sense. What we have argued is that 25 *cents* is the same amount of money as one *quarter*, and therefore (1 *quarter*/25 *cents*) = 1.

2.5.3 Guidelines for Calculating with Units

The following set of guidelines helps when calculating with units.

Guideline 1. Before inserting a number into a formula, convert it to a number having the proper units, so that the units being dealt with will cancel. For example, use $t = D/S$ to calculate time from distance and speed, where speed is equal to 0.4 km/sec. If you are given that the distance is 800 meters (800 m), you must convert this distance into kilometers before inserting it into the formula. This is done by:

$$800 \text{ m} \left(\frac{1 \text{ km}}{1000 \text{ m}}\right)_{CF} = 0.8 \text{ km}$$

Guideline 2. After inserting numbers into a formula, the unit symbols may be cancelled as if they were ordinary numbers. To continue the example, then

$$t = \frac{0.8 \text{ km}}{0.4 \text{ km/sec}} = 2.0 \text{sec}$$

Guideline 3. Always write the units explicitly at every step of the calculation, including in the final answer, to avoid making mistakes or giving meaningless numbers as answers. For example, if someone asks you how far is it from your hometown to Antarctica, it is meaningless to say "six thousand," unless you also say what units this is being expressed in.

Guideline 4. After calculating (by hand or using a calculator) a final number, you need to decide how many significant digits to write when giving your answer to a problem. This determines what *precision* you will use in stating your answer. For example, the answer 2.0 seconds in the above problem is given using two digits: 2 and 0. This means that you believe the answer is not as large as 2.1 seconds, but is not as small as 1.9 seconds. Strictly speaking, it means that the answer is somewhere between 1.95 seconds and 2.05 seconds. If the data you used to calculate your answer are not actually known to this degree of precision, you should perhaps report your answer using only one digit. This implies a lower precision of your answer. For example, the answer 2.0 seconds could be reported instead as 2 seconds, which means that the answer is somewhere between 1.5 seconds and 2.5 seconds.

QUICK QUESTION 2.3

Use the following conversion factors to calculate the number of days in one century: 1 century = 100 years; 1 year = 52 weeks; 1 week = 7 days. Now repeat, using 1 year = 12 months; 1 month = 4 weeks; 1 week = 7 days. Again repeat, using 1 year = 365 days. Explain why each answer is different, though "correct," given the precision that each calculation is using.

2.6 PROPORTIONALITY

The simplest relation between two mathematical quantities is that of direct proportionality, meaning that if one quantity increases by a certain multiplying factor, the other increases by the same multiplying factor. For example, if you have a telephone billing plan that charges strictly by the amount of time you spend on the phone, then the cost of a call is proportional to the time connected. If the cost rate is 0.1 cents per second, then we can express the total cost by the equation

$$\text{total cost} = (0.1 \text{ cents per second}) \times (\text{time connected})$$

We say that the total cost is proportional to the time connected, and symbolize this by

$$\text{total cost} \propto \text{time connected}$$

Say that you talk for 100 seconds. This would cost 10 cents. If you double the time connected, the cost would double to 20 cents, etc.

Another kind of proportionality is inverse proportionality, meaning that if one quantity is increased by multiplying with a certain factor, the other decreases by dividing by the same factor. For example, in a long concert hall, the loudness of the music might be inversely proportional to your distance from the stage. If you double your distance, the loudness would be cut in half. That is,

$$\text{perceived loudness} \propto \frac{\text{loudness at 1 foot from stage}}{\text{distance in feet from stage}}$$